

Solitary and compact-like shear waves in the bulk of solids

Michel Destrade*

*Laboratoire de Modélisation en Mécanique UMR 7607, CNRS,
Université Pierre et Marie Curie, 4 place Jussieu, case 162, 75252 Paris Cedex 05, France*

Giuseppe Saccomandi†

*Sezione di Ingegneria Industriale, Dipartimento di Ingegneria dell'Innovazione,
Università degli Studi di Lecce, 73100 Lecce, Italy.*

(Dated: February 8, 2008)

We show that a model proposed by Rubin, Rosenau, and Gottlieb [J. Appl. Phys. **77** (1995) 4054], for dispersion caused by an inherent material characteristic length, belongs to the class of simple materials. Therefore, it is possible to generalize the idea of Rubin, Rosenau, and Gottlieb to include a wide range of material models, from nonlinear elasticity to turbulence. Using this insight, we are able to fine-tune nonlinear and dispersive effects in the theory of nonlinear elasticity in order to generate pulse solitary waves and also bulk travelling waves with compact support.

PACS numbers: 46.83.10, 05.45.Yv

The emergence of significant solitary waves from the equations governing the motion of solids requires usually a fine balance between *nonlinearity* and *dispersion*. The first ingredient finds its natural source in the theory of finite, fully nonlinear, elasticity, but the source for the second is more elusive. Often, specific boundary conditions (such as: a thin film coated on a substrate, the finite interface between phases in an elastic composite material, the lateral dimensions of a rod with free surfaces, etc.) are used because they introduce a characteristic length, to be compared with the wavelength. For homogeneous infinite (bulk) solids, the presence of a microstructure is invoked, recording the presence of inherent material characteristic lengths due to, for instance: the atomic lattice cells in a metal, the grain size in an alloy, the persistence and contour length in a polymer, etc. (it is worth noting that a similar situation also occurs in fluid dynamics, when inherent characteristic lengths may be not ignored in models of turbulence.) Although the standard constitutive equations of continuum mechanics can accommodate large strains and nonlinear material responses, they do not in general include the effects of characteristic material lengths [1]. The introduction of such information in models of material behavior is a subtle and difficult problem [2], and the modifications of the basic constitutive equations proposed in the literature are often unsatisfactory because they lead to physically unrealistic and mathematically ill-posed models [1, 3]. Finally, finding a simple connection between nonlinear and dispersive effects in material modeling is an interesting problem *per se*, because the balance between these two effects is at the origin of many fascinating phenomena, and in particular the propagation of solitons and compactons [4].

In 1987 Rosenau [5] proposed a regularized functional expansion of the equations for the dynamics of dense lattices in order to yield the leading effect of dispersion due to a characteristic length, via a physically rea-

sonable and mathematically tractable differential equation. One of the advantages of this equation over other micro-mechanical or nonlocal approaches is that it models dispersion without increasing the number of initial or boundary conditions of the usual wave equation. On the same pathway, a three-dimensional phenomenological continuum model of the Rosenau approach was proposed by Rubin, Rosenau, and Gottlieb (RRG) in 1995 [3]. One aim of this Letter is to put in perspective that latter model by showing that it is a special case of the theory of simple materials in the sense of Noll [6], hence generalizing the original idea of Rosenau to a large class of material behavior, from nonlinear elasticity to fluid turbulence. Another objective is to find a clear and direct way to balance nonlinearity and dispersion in order to generate solitary waves with compact support in solids. This balance is attained by relying solely on constitutive arguments, without any resort to reductive asymptotic expansions; it is then compared (favorably, as it turns out) with the results obtained by such methods.

Let the motion of a body be described by a relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where \mathbf{x} denotes the current coordinates of a point occupied by the particle of coordinates \mathbf{X} in the reference configuration at time t . The deformation gradient $\mathbf{F}(\mathbf{X}, t)$ and the spatial velocity gradient $\mathbf{L}(\mathbf{X}, t)$ associated with the motion are defined by $\mathbf{F} = \text{Grad } \mathbf{X}$ and $\mathbf{L} = \text{grad } \mathbf{v}$ ($\mathbf{v} = \partial \mathbf{x} / \partial t$), respectively (in this Letter, grad and div denote respectively the gradient and divergence operators with respect to the current position \mathbf{x} , and Grad and Div denote the corresponding operators with respect to \mathbf{X} .) The mathematical theory of simple materials is well-grounded from the mathematical and thermomechanical points of view [6]; there, the Cauchy stress \mathbf{T} is determined by the whole history of the deformation gradient,

$$\mathbf{T} = \mathcal{G}_{s=0}^{\infty} (\mathbf{F}(t-s)), \quad (1)$$

where \mathcal{G} is the constitutive functional. We now summarize the RRG model, which itself follows the axiomatic approach to continuum mechanics introduced by Green and Naghdi [8]. Let us introduce the specific entropy per unit mass η , the specific rate of internal production of entropy ξ , the entropy flux per unit of present area \mathbf{p} , and the specific Helmholtz free energy ψ . The set $\{\eta, \xi, \mathbf{p}, \psi, \mathbf{T}\}$ must satisfy, for all possible thermomechanical process, the restrictions coming from the balance of angular momentum, the energy equation, and from a suitable statement of the second law of thermodynamics. The central idea of Rubin et al. [3] is to add terms $\{\psi_2, \mathbf{T}_2\}$, *modeling material dispersion*, to the Cauchy stress tensor and the free energy: $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$, $\psi = \psi_1 + \psi_2$, say, while the other quantities $\{\eta, \xi, \mathbf{p}\}$ and the associated constitutive equations stay unchanged. They show that the balance of angular momentum and the energy equation then lead to $\mathbf{T}_2 = \mathbf{T}_2^T$ and

$$\rho \dot{\psi}_2 - \mathbf{T}_2 \cdot \mathbf{D} = 0, \quad (2)$$

respectively, where ρ is the mass density per unit volume, a superposed dot denotes the material time derivative (at \mathbf{X} fixed), and $\mathbf{D} = (\mathbf{L} + \mathbf{L}^T)/2$ is the stretching tensor. Further, the RRG model considers that ψ_2 is a single-valued function: $\psi_2 = \psi_2(\delta)$, where $\delta \equiv \mathbf{D} \cdot \mathbf{D}$ is an isotropic invariant; then from (2) follows that

$$\mathbf{T}_2 = \rho \psi_2'(\delta) [\text{grad } \dot{\mathbf{v}} + (\text{grad } \dot{\mathbf{v}})^T + 2\mathbf{L}^T \mathbf{L} - 4\mathbf{D}^2]. \quad (3)$$

Our first result is to show that \mathbf{T}_2 is a special case of the Cauchy stress $\boldsymbol{\sigma}$ associated with those non-Newtonian simple fluids denoted in the literature as *second-grade fluids* [6, 7],

$$\boldsymbol{\sigma} = \nu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \quad (4)$$

where ν is the classical Newtonian viscosity, α_1, α_2 are the microstructural coefficients (which may be taken as constants or as functions, for example of δ), and the first two Rivlin-Ericksen tensors $\mathbf{A}_1, \mathbf{A}_2$ are defined by: $\mathbf{A}_1 = 2\mathbf{D}$, $\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^T \mathbf{A}_1$. Indeed, recall that $\mathbf{L} = (\text{Grad } \mathbf{v})\mathbf{F}^{-1} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, from which follows that

$$\dot{\mathbf{L}} = (\text{Grad } \dot{\mathbf{v}})\mathbf{F}^{-1} + \mathbf{L}\mathbf{F}(\mathbf{F}^{-1}) = \text{grad } \dot{\mathbf{v}} - \mathbf{L}^2, \quad (5)$$

(where the last equality comes from the material time derivative of the identity $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$). Substituting into $\dot{\mathbf{A}}_1 = \dot{\mathbf{L}} + \dot{\mathbf{L}}^T$ and into the definition of \mathbf{A}_2 , we find that the tensors (3) and (4) coincide when $\nu = 0$ and $\alpha_1(\geq 0) = -\alpha_2 = \rho \psi_2'$. We conclude that when \mathbf{T}_1 is in the material class (1) then the total Cauchy stress tensor $\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2$ of the corresponding RRG model is also that of a simple material. This observation has many interesting consequences. For example, if p is the Lagrange multiplier associated with the constraint of incompressibility ($\det \mathbf{F} = 1, \text{tr } \mathbf{D} = 0$), then the classical choice of the Navier-Stokes stress tensor $\mathbf{T}_1 = -p\mathbf{1} + \nu \mathbf{D}$ for

the RRG model leads to those incompressible homogeneous fluids of second grade for which $\alpha_1 + \alpha_2 = 0$. We note that the equations of motion associated with those fluids, in the general and some special case, have been found [9] to coincide with the equations derived by other means (asymptotic expansions, Lagrangian means, average Euler equation, etc.) in several models of turbulence and shallow water theory [10]. It is now clear that the presence of the RRG characteristic length is the reason of such connections.

Let us consider the case of the RRG model when \mathbf{T}_1 corresponds to a hyperelastic, incompressible, isotropic solid [6],

$$\mathbf{T}_1 = -p\mathbf{1} + 2(\partial \Sigma / \partial I_1) \mathbf{B} - 2(\partial \Sigma / \partial I_2) \mathbf{B}^{-1}. \quad (6)$$

Here $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, and $\Sigma = \Sigma(I_1, I_2)$ is the strain energy density, a function of the invariants $I_1 = \text{tr } \mathbf{B}$ and $I_2 = \text{tr } \mathbf{B}^{-1}$. Then the RRG model is a special case of the so-called *solid of second grade* [11, 12]. A rectilinear shearing motion is defined by

$$\mathbf{x} = [X + u(Y, t)] \mathbf{e}_1 + Y \mathbf{e}_2 + Z \mathbf{e}_3, \quad (7)$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a fixed orthonormal triad. Then $\mathbf{B} = \mathbf{1} + u_Y^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + u_Y(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, $\mathbf{B}^{-1} = \mathbf{1} + u_Y^2 \mathbf{e}_2 \otimes \mathbf{e}_2 - u_Y(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, $\mathbf{A}_1 = u_{Yt}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$, and $\mathbf{A}_2 = u_{Ytt}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + 2u_{Yt}^2 \mathbf{e}_2 \otimes \mathbf{e}_2$. Also, $I_1 = I_2 = 3 + u_Y^2$ and $\delta = u_{Yt}^2$. The equations of motion in the absence of body forces ($\text{div } \mathbf{T} = \rho \dot{\mathbf{v}}$) reduce to the three scalar equations [12]:

$$-\frac{\partial p}{\partial X} + \frac{\partial T_{12}}{\partial Y} = \rho \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial T_{22}}{\partial Y} = 0, \quad -\frac{\partial p}{\partial Z} = 0, \quad (8)$$

for which we used (3), or its equivalent form: $\mathbf{T}_2 = \rho \psi_2'(\mathbf{A}_2 - \mathbf{A}_1^2)$. The third equation gives $p = p(X, Y, t)$ and then the cross-differentiation of the first and second equations leads to $p(X, Y, t) = -2\Sigma_2 u_Y^2 + \rho \psi_2' u_{Yt}^2 + p_1(t)X + p_0(t)$, where p_0, p_1 are arbitrary functions of t alone and Σ_i is the partial derivative of Σ with respect to I_i . We are left with a single determining equation for the displacement $u(Y, t)$: $p_1 + [2(\Sigma_1 + \Sigma_2)u_Y + \rho \psi_2' u_{Ytt}]_Y = \rho u_{tt}$ or, taking the derivative with respect to Y , a single equation for the *strain* $U \equiv u_Y$,

$$[\hat{\mu}U + \hat{\beta}U_{tt}]_{YY} = \rho U_{tt}, \quad \hat{\mu} = 2(\Sigma_1 + \Sigma_2), \quad \hat{\beta} = \rho \psi_2'. \quad (9)$$

Here $\hat{\mu} = \hat{\mu}(U^2) > 0$ is the generalized shear modulus of nonlinear elasticity [13, 14]; it is a constant (Lamé's second constant) in linear elasticity, and also in nonlinear elasticity for the special cases of the neo-Hookean ("geometric" nonlinearity) and of the Mooney-Rivlin ("geometric" and "physical" nonlinearities) forms of the strain energy $W (= C_{10}(I_1 - 3), C_{10}(I_1 - 3) + C_{01}(I_2 - 3)$, respectively). However in general, a nonlinear constitutive equation for the solid yields a non-constant $\hat{\mu}$; similarly,

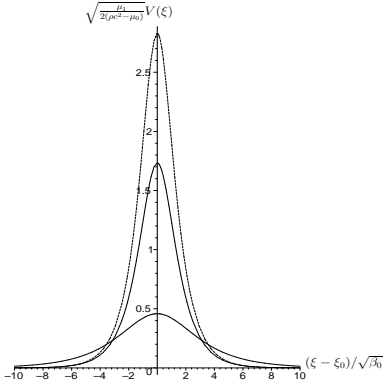


FIG. 1: Profile of the solitary wave when $\rho c^2/\mu_0 = 1.1$ (thick curve), 2 (thin curve), 3 (dashed curve).

a general dispersive material behavior is attested by a non-constant function $\hat{\beta} = \hat{\beta}(U_t^2) > 0$ (the case where $\hat{\mu}$ and $\hat{\beta}$ are both constants is considered elsewhere [12]). To illustrate some features of the RRG model, we make the following constitutive choices: $\hat{\mu}(U_t^2) = \mu_0 + \mu_1 U_t^2$ and $\hat{\alpha}(U_t^2) = \rho(\beta_0 + \beta_1 U_t^2)$, where $\mu_0 > 0$ is the infinitesimal shear modulus of the solid. We seek a travelling wave solution: $U = V(\xi)$, $\xi = Y - ct$, with drop boundary conditions: $V^{(n)}(\pm\infty) = 0$, $n \geq 0$. Then successive integrations of the reduced governing equation lead to

$$\rho c^4 \beta_1 (V')^4 + 2\rho c^2 \beta_0 (V')^2 + [\mu_1 V^2 - 2(\rho c^2 - \mu_0)]V^2 = 0. \quad (10)$$

For a linear variation of the dispersion free energy ψ_2 with the isotropic invariant $\delta = \mathbf{D} \cdot \mathbf{D}$, we have $\beta_1 = 0$ and the following *solitary wave* solution:

$$V(\xi) = \sqrt{2\frac{\rho c^2 - \mu_0}{\mu_1}} \operatorname{sech} \sqrt{\frac{\rho c^2 - \mu_0}{\rho c^2} \frac{\xi - \xi_0}{\sqrt{\beta_0}}}, \quad (11)$$

where ξ_0 is arbitrary. This wave is supersonic with respect to the speed of an infinitesimal shear wave but c is otherwise arbitrary. Figure 1 shows its profile for several values of $\rho c^2/\mu_0$.

Now, we know from previous results [15] that (10) has no compact-like solutions when $\beta_0 \beta_1 \neq 0$. On the other hand, when $\beta_0 = 0$, $\beta_1 \neq 0$, it is of the form

$$(V')^4 = (\gamma_0 - \gamma_1 V^2)V^2, \quad (12)$$

(where $\gamma_0 = 2(\rho c^2 - \mu_0)/(\rho c^4 \beta_1)$ and $\gamma_1 = \mu_1/(\rho c^4 \beta_1)$), the right hand-side of which has a double zero (at $V = 0$) and two simple zeros (at $V = \pm\sqrt{\gamma_0/\gamma_1}$). This observation alone is sufficient to conclude that (12) admits a *compact wave* solution. As it happens, (12) can be integrated in terms of special functions. Indeed, introducing the hypergeometric function $\operatorname{Hyp}_2F_1[a, b, c, z]$, solution to the second-order linear differential equation $z(1-z)y'' + [c - (a+b+1)z]y' - aby = 0$, we find the

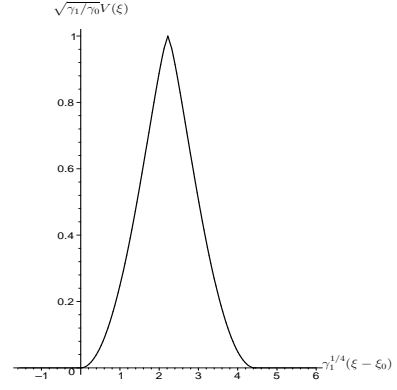


FIG. 2: Profile of the compact wave.

following identity,

$$\int (u^2 - u^4)^{-1/4} du = 2(u^2)^{1/4} \operatorname{Hyp}_2F_1 \left[\frac{1}{4}, \frac{1}{4}, \frac{5}{4}, u^2 \right]. \quad (13)$$

We call $\mathcal{I}(u)$ that function and observe that it is defined on the interval $0 \leq u \leq \pi/\sqrt{2}$, where it increases in a monotone way from 0 to 1. Then we manipulate (12) to give

$$\sqrt{\gamma_1/\gamma_0} V(\xi) = \mathcal{I}^{-1}[\gamma_1^{1/4}(\xi - \xi_0)], \quad (14)$$

where ξ_0 is arbitrary. Hence we can build a *weak* solution with *finite* support measure $\sqrt{2}\pi\gamma_1^{-1/4}$, defined by:

$$\begin{aligned} \sqrt{\gamma_1/\gamma_0} V(\xi) &= \mathcal{I}^{-1}(\gamma_1^{1/4} \xi) && \text{on } [0, \pi/a], \\ &= \mathcal{I}^{-1}[\gamma_1^{1/4}(\sqrt{2}\pi - \xi)] && \text{on } [\pi/a, \sqrt{2}\pi/a], \end{aligned} \quad (15)$$

and zero *everywhere else*, see Figure 2. (Here $a = (4\gamma_1)^{1/4}$). This wave shares some characteristics with the solitary wave (11): it is supersonic, its amplitude is proportional to $\sqrt{\gamma_0/\gamma_1} = \sqrt{2(\rho c^2 - \mu_0)/\mu_1}$, growing with the speed; in contrast, its width evolves as $\gamma_1^{-1/4}$, which is proportional to c . We emphasize that this compact wave is derived solely on the basis of constitutive arguments, by taking the generalized shear modulus $\hat{\mu}$ linear in the squared shear (nonlinear effect) and the free energy density ψ_2 proportional to δ^2 (dispersive effect). Previous attempts at finding compact waves relied on ad hoc asymptotic reductive approaches [16] and to the best of our knowledge, a bulk compact wave has never been generated in the framework of nonlinear elasticity [1, p. 91]

Finally we look at reductive asymptotic expansions of the strain equation of motion (9). We consider that the generalized shear modulus is slightly nonlinear, $\hat{\mu} = \mu_0(1 + \epsilon U^2)$ where ϵ is a small parameter, and that the dispersion is also of order ϵ : for instance, when ψ_2 is linear in δ , we take: $\hat{\beta} = 2(\rho a^2)\epsilon$, where a is a constant with the dimensions of a length. With the stan-

dard semi-characteristic variable stretching transformation [17]: $U(Y, t) = v(\eta, \epsilon\tau)$, where $\tau = [\mu_0/(\rho a^2)]^{1/2}t$ and $\eta = Y/a - \tau$ are dimensionless variables, we find that at order ϵ , (9) is the modified KdV equation,

$$v_\tau + \frac{1}{2}(v^3)_\eta + v_{\eta\eta\eta} = 0, \quad (16)$$

for which the solitonic solution is a *sech*-wave as in (11). When ψ_2 is proportional to δ^2 , we take $\hat{\beta}$ in the form $\hat{\beta} = 2(\rho^2 a^4/\mu_0)\epsilon U_t^2$, and find

$$v_\tau + \frac{1}{2}(v^3)_\eta + \frac{1}{3}(v^3)_{\eta\eta} = 0, \quad (17)$$

an equation similar to the $K(3, 3)$ KdV equation [4, 18]. Although (16) and (17) might have rich implications, we do not study them further because the generality of (9) has clearly been lost in the process of deriving them. In particular, the traveling equation (10) is just as rewarding, and is obtained exactly, without the need for a time-stretching variable, a moving frame, and an asymptotic expansion where higher-order terms are simply ignored.

In conclusion, we believe that this investigation demonstrates the usefulness, the versatility, and ultimately the beauty of the RRG model. Although the RRG model originates from a microstructural model [5], it can now be apprehended as a purely phenomenological approach to dispersion, applicable to the whole class of simple materials. It overcomes the usual problems of nonlocal theories, it unifies several results in the literature, and it provides a natural and detailed study of localization of traveling waves in elastic materials. Yet several topics raised here merit further scrutiny: is the solitary wave exhibited a *soliton*? is the compact wave a *compacton*? Also, the analysis relied on *specific* constitutive equations. Hence for the nonlinear elasticity, the generalized shear modulus was that of a solid having strain energy density in the form [14] $\Sigma = C_{10}J_1 + C_{01}J_2 + C_{20}J_1^2 + C_{11}J_1J_2 + C_{02}J_2^2$ where $J_1 = I_1 - 3$ and $J_2 = I_2 - 3$ (so that $\mu_0 = 2(C_{10} + C_{01})$ and $\mu_1 = 4(C_{20} + C_{11} + C_{02})$). Although this, and higher-order, truncation of the polynomial expansion of Σ are highly popular in experimental nonlinear elasticity, in curve fitting, and in finite element packages, it should be acknowledged that the determination of the “best” C_{ij} is a messy procedure [20]. Eventually, it might be of greater (and safer) interest to consider other types of nonlinear strain energy densities, such as those arising in biological soft tissues, which incorporate the effect of strain hardening. Similar remarks hold for the non-

linear dispersion, for which the free energy was taken as $\psi_2 = \alpha_0\delta + (\alpha_1/2)\delta^2$.

This work was supported by a Séjour Scientifique de Haut Niveau from the French Ministère des Affaires Étrangères, by GNFM and by MIUR PRIN 2004.

* Electronic address: destrade@lmm.jussieu.fr

† Electronic address: giuseppe.saccomandi@unile.it

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